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1996 J. Phys. A: Math. Gen. 29 4275

(http://iopscience.iop.org/0305-4470/29/14/043)

J. Phys. A: Math. Gen. **29** (1996) 4275–4288. Printed in the UK

The angle-angular momentum quantum phase space

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Received 23 January 1996

Abstract. The angle-angular momentum quantum phase space is considered and the corresponding Heisenberg–Weyl group $\mathcal{D}$ is studied. When $(2j+1)$ is a power of a prime, $Z(2j+1)$ is a Galois field and stronger results can be proved. $SL(2, Z(2j+1))$ transformations are explicitly constructed and various properties of the displacement operators are studied. Central extensions of the Abelian group $G = \mathcal{D}|Z(2j+1)$ by $Z(2j+1)$ are studied and they provide all the ways of constructing the Heisenberg–Weyl group $\mathcal{D}$ from G and $Z(2j+1)$.

1. Introduction

The study of finite quantum systems was initiated a long time ago [1]. More recently this work has been extended by various authors [2] in various contexts. Work on finite Fourier transforms [3, 4] is also related to this topic.

In [5] we have studied various aspects of the angle-angular momentum quantum phase space. Performing a finite Fourier transform on the usual angular momentum states (which we denote as $|J; jm\rangle$) we introduce the dual angle states (which we denote as $|\theta; jm\rangle$). Performing the same Fourier transform on both sides of the angular momentum operators (J_+, J_-, J_z) we obtain the angle operators $(\theta_+, \theta_-, \theta_z)$ which form an $SU(2)$ algebra. In a similar way as $|J; jm\rangle$ are eigenstates of J^2, J_z, $|\theta; jm\rangle$ are eigenstates of θ^2, θ_z. We have also introduced displacement operators and the corresponding Heisenberg–Weyl group. The next step is to study $SL(2, Z(2j+1))$ transformations in the angle-angular momentum quantum phase space. They are the analogue of the $SL(2, R)$ transformations in the harmonic oscillator context, which lead to Bogoliubov transformations. Important special cases of the $SL(2, Z(2j+1))$ transformations have been explicitly constructed. We have also used the Chinese remainder theorem to study a factorization of the $(2j+1)$-dimensional Hilbert space of our system in terms of other smaller Hilbert spaces of subsystems. We have shown that every state of the original system can be expressed as a product of states of the subsystems and every operator acting on the original system can be expressed as a product of operators which act on the subsystems. In this sense the quantum mechanics of the original system is factorized into quantum mechanics in smaller subsystems.

In this paper we extend these ideas further. In section 2 we present the basic definitions and prove some new relations which complement our previous work. In sections 3 and 4 we show that when $(2j+1)$ is a power of a prime stronger results can be derived. The reason for this is that in our calculations we use integers in $Z(2j+1)$ (the integers modulo $2j+1$). This is, in general, a commutative ring with a unity and, in the case when $(2j+1)$ is a power of a prime, a Galois field. The existence of inverses in the Galois case leads to stronger results. These stronger results are also indirectly applicable to non-Galois systems,

0305-4470/96/144275+14$19.50

because they can be factorized in terms of Galois subsystems. In section 3 we construct explicitly the operators that perform general $SL(2, Z(2j+1))$ transformations on Galois angular momentum systems. In section 4 we discuss the displacement operators and show that they are intimately related to the Wigner function.

Let $\mathcal{D}$ be the Heisenberg–Weyl group for the angle-angular momentum, $Z(2j+1)$ its centre, and G the Abelian group $G = \mathcal{D}/Z(2j+1)$ that performs displacements in the angle-angular momentum phase space. In section 5 we consider the deeper question of whether there are many Heisenberg–Weyl groups that we can build from the Abelian group G and $Z(2j+1)$. Mathematically, this is the problem of central extensions of G by $Z(2j+1)$ [6, 7]. In the context of the magnetic translation groups, central extensions have been studied in [8]. We have also used central extensions in the context of gauge theories in [9].

We conclude in section 6 with a discussion of our results. In the appendix we give some formulae which are useful in calculations of matrix elements of the operators J_z, θ_z.

2. The J_z–θ_z phase space: a discretized torus

As in our previous work on this problem [5], we denote by $|J; jm\rangle$ the usual angular momentum states. m belongs to $Z(2j+1)$ (the integers modulo $2j+1$). We have shown that the various formulae are slightly different in the bosonic case (integer j) from those in the fermionic case (half-integer j) and in this paper we limit our discussion to the bosonic case only. The states $|J; jm\rangle$ span the Hilbert space $H(2j+1)$. In [5] we have considered the finite Fourier transform

$$U_F = (2j+1)^{-1/2} \sum_{m,n} \omega(mn) |J; jm\rangle\langle J; jn| \tag{2.1}$$

$$\omega(\alpha) = \exp\left[\mathrm{i}\frac{2\pi\alpha}{2j+1}\right] \tag{2.2}$$

$$U_F U_F^+ = U_F^+ U_F = \mathbf{1} \tag{2.3}$$

$$U_F^4 = \mathbf{1}. \tag{2.4}$$

Using these Fourier transforms we have introduced the θ-basis of Euler angle states $|\theta; jm\rangle$ dual to the usual J-basis of angular momentum states $|J; jm\rangle$:

$$|\theta; jm\rangle = U_F |J; jm\rangle = (2j+1)^{-1/2} \sum_{n=-j}^{j} \omega(mn) |J; jn\rangle. \tag{2.5}$$

We have also introduced the Euler angle operators θ_+, θ_-, θ_z which obey the $SU(2)$ algebra:

$$\theta_z = U_F J_z U_F^+ \tag{2.6}$$

$$\theta_+ = U_F J_+ U_F^+ \tag{2.7}$$

$$\theta_- = U_F J_- U_F^+ \tag{2.8}$$

$$[\theta_z, \theta_\pm] = \pm\theta_\pm \qquad [\theta_+, \theta_-] = 2\theta_z. \tag{2.9}$$

The θ-operators act on the θ-states in an analogous way to the J-operators acting on the J-states. We have also considered the J_z–θ_z phase space which is the discretized torus,

$$T = Z(2j+1) \times Z(2j+1) \tag{2.10}$$

and we have introduced the unitary 'ladder operators':

$$E = \exp\left[-\mathrm{i}\frac{2\pi}{2j+1}\theta_z\right] \tag{2.11}$$

$$F = \exp\left[\mathrm{i}\frac{2\pi}{2j+1}J_z\right] \tag{2.12}$$

$$E^{2j+1} = F^{2j+1} = 1 \tag{2.13}$$

$$E^{\beta}F^{\alpha} = F^{\alpha}E^{\beta}\omega(-\alpha\beta) \qquad \alpha, \beta \in Z(2j+1) \tag{2.14}$$

which perform displacements along the J_z and θ_z axes, correspondingly:

$$E^{\beta}|J; jm\rangle = |J; jm+\beta\rangle \tag{2.15}$$

$$E^{\beta}|\theta; jm\rangle = \omega(-\beta m)|\theta; jm\rangle \tag{2.16}$$

$$F^{\alpha}|J; jm\rangle = \omega(m\alpha)|J; jm\rangle \tag{2.17}$$

$$F^{\alpha}|\theta; jm\rangle = |\theta; jm+\alpha\rangle. \tag{2.18}$$

Note that

$$U_F E U_F^{+} = F \tag{2.19}$$

$$U_F F U_F^{+} = E^{+} \tag{2.20}$$

$$E^{k} J_Z E^{-k} = J_Z - k\mathbf{1} \tag{2.21}$$

$$E^{k}\theta_z E^{-k} = \theta_Z \tag{2.22}$$

$$F^{k} J_Z F^{-k} = J_Z \tag{2.23}$$

$$F^{k}\theta_z F^{-k} = \theta_z - k\mathbf{1}. \tag{2.24}$$

Successive action of the Fourier operator U_F on the states $|J; jm\rangle$ and $|\theta; jm\rangle$ gives

$$|J; jm\rangle \stackrel{U_F}{\to} |\theta; jm\rangle \stackrel{U_F}{\to} |J; j-m\rangle \stackrel{U_F}{\to} |\theta; j-m\rangle \stackrel{U_F}{\to} |J; jm\rangle. \tag{2.25}$$

Successive action of the operators U_F, U_F^{+} on the left and right of the operators J_z, θ_z and E, F gives

$$J_z \stackrel{U_F}{\to} \theta_z \stackrel{U_F}{\to} -J_z \stackrel{U_F}{\to} -\theta_z \stackrel{U_F}{\to} J_z \tag{2.26}$$

$$E \stackrel{U_F}{\to} F \stackrel{U_F}{\to} E^{+} \stackrel{U_F}{\to} F^{+} \stackrel{U_F}{\to} E. \tag{2.27}$$

The J_z, θ_z are finite-dimensional matrices and therefore their powers are not all independent. Using the Cayley–Hamilton theorem of the theory of matrices we can express the $(2j+1)$-power of these operators as a linear combination of the lower powers. The eigenvalues of θ_z are the integers from $-j$ to j and the characteristic polynomial is

$$P(x) = x\prod_{m=1}^{j}(x^2 - m^2) = x^{2j+1} + \mu_{2j-1}x^{2j-1} + \cdots + \mu_1 x \tag{2.28}$$

where the above equation defines the integers μ_i. Note that only odd powers appear in this polynomial. The Cayley–Hamilton theorem states that

$$P(\theta_z) = \theta_z^{2j+1} + \mu_{2j-1}\theta_z^{2j-1} + \cdots + \mu_1\theta_z = 0. \tag{2.29}$$

This implies that an ‘arbitrary’ function $f(\theta_z)$ is defined modulo the polynomial $P(\theta_z)$ and therefore it can be reduced to the ‘remainder’ polynomial $R(\theta_z)$ of order $2j$. The coefficients of $R(\theta_z)$ can be calculated as follows:

$$\begin{aligned} f(x) &= P(x)Q(x) + R(x) \\ R(x) &= \sum_{\mu=0}^{2j} a_\mu x^{\mu}. \end{aligned} \tag{2.30}$$

We insert in this equation the roots of $P(x)$ and obtain a system of $(2j+1)$ equations with $(2j+1)$ unknowns:

$$\sum_{\mu=0}^{2j} a_\mu m^\mu = f(m) \qquad m = 0, \pm 1, \ldots, \pm j. \tag{2.31}$$

In this way the function $f(\theta_z)$ simplifies to the polynomial $R(\theta_z)$. It is clear that only the first $2j$ powers of θ_z are independent and all the higher powers are linear combinations of them. Similar results hold for J_z which also has $P(x)$ as a characteristic polynomial.

Note that the operators J_z, θ_z are defined modulo an integer multiple of $(2j+1)\mathbf{1}$. Indeed, in the equation

$$J_z = \sum_{m=-j}^{j} m|J; jm\rangle\langle J; jm| \tag{2.32}$$

m is defined modulo $(2j+1)$. If we replace m by $m + v(2j+1)$ we obtain the operator $J_z + v(2j+1)\mathbf{1}$. An alternative way of seeing this is through the transformations

$$[W_J(v)]J_z[W_J(v)]^+ = J_z + v(2j+1)\mathbf{1} \tag{2.33}$$

where

$$W_J(v) \equiv E^{v(2j+1)} = \mathbf{1} \tag{2.34}$$

can be viewed as a winding operator on the 'J_z-circle' of the toroidal J_z–θ_z phase space.

A similar argument can be given for θ_z in terms of the winding operator

$$W_\theta(w) \equiv F^{w(2j+1)} = \mathbf{1} \tag{2.35}$$

on the θ_z-circle of the toroidal J_z–θ_z phase space. Therefore we write symbolically

$$J_z = J_z + v(2j+1)\mathbf{1}(\mathrm{mod}(2j+1)\mathbf{1}) \tag{2.36}$$

$$\theta_z = \theta_z + w(2j+1)\mathbf{1}(\mathrm{mod}(2j+1)\mathbf{1}) \tag{2.37}$$

where v, w are integers which can be viewed as winding numbers.

3. Galois quantum systems and $SL(2, Z(2j+1))$ transformations

The above results are valid for any integer j. However, stronger results can be obtained when $(2j+1)$ is a power of a prime. In this section we first clarify this point and then discuss the $SL(2, Z(2j+1))$ transformations.

When $(2j+1)$ is not a power of a prime p

$$2j+1 \neq p^m \tag{3.1}$$

the $Z(2j+1)$ is a commutative ring with a unity, which is not a field. Only when $(2j+1)$ is a power of a prime p,

$$2j+1 = p^m \tag{3.2}$$

is $Z(p^m)$ a field. This is a famous result by Galois and the corresponding fields are called Galois fields. For $m = 1$ it is not difficult to see that $Z(p)$ is a field. For $m > 1$ the $Z(p^m)$ is a field extension of $Z(p)$, of degree m. Its elements can be written as polynomials of an 'indeterminate' x with coefficients in $Z(p)$. These polynomials are defined modulo an irreducible polynomial $p(x)$ of degree m. The use of different irreducible polynomials of the same degree m leads to finite fields $Z(p^m)$ which are isomorphic to each other. In this sense there is only one finite field $Z(p^m)$. Addition and multiplication tables for $Z(p^m)$

are, in general, difficult to construct, but for practical applications they can be found in tables. The notation $GF(p^m)$ is also used for $Z(p^m)$. The advantages of having a field are related to the fact that in a field all non-zero elements have an inverse which is in contrast to commutative rings with a unity where the inverse of a non-zero element might or might not exist. We call Galois quantum systems finite quantum systems with a Hilbert space whose dimension is the power of a prime (equation (3.2)). In the following we will make clear which results are general and which are stronger results valid only for Galois systems.

At this point it is relevant to point out that in [5] we have studied the factorization of a finite system into subsystems. If the dimension $(2j+1)$ of the Hilbert space $H(2j+1)$ of the system can be factorized as

$$2j+1 = (2j_1+1) \times \cdots \times (2j_N+1) \tag{3.3}$$

where any two of these factors are coprime, then we have shown that there exists a unitary isomorphism between $H(2j+1)$ and the product of Hilbert spaces $H(2j_\lambda+1)(\lambda = 1, \ldots, N)$ with dimension $(2j_\lambda+1)$:

$$H(2j+1) = H(2j_1+1) \times \cdots \times H(2j_N+1). \tag{3.4}$$

Every state in $H(2j+1)$ is expressed as a product of states in the various $H(2j_\lambda+1)$ and every operator acting upon $H(2j+1)$ is expressed as a product of operators acting upon the various $H(2j_\lambda+1)$. This is very important for our purposes because we can factorize in a unique way an arbitrary $(2j+1)$ as

$$2j+1 = p_1^{n_1} \ldots p_N^{n_N} \tag{3.5}$$

and, therefore, any angular momentum quantum system can be viewed as a product of Galois angular momentum subsystems. In this sense all our results concerning Galois angular momentum systems, through the above factorization, become relevant to all angular momentum systems.

We now consider the transformations

$$\begin{aligned} E &\to E' = E^\alpha F^\beta \\ F &\to F' = E^\gamma F^\delta \end{aligned} \tag{3.6}$$

where $\alpha, \beta, \gamma, \delta$ are integers in $Z(2j+1)$ such that

$$\alpha\delta - \beta\gamma = 1(\mathrm{mod}(2j+1)). \tag{3.7}$$

These transformations preserve equations (2.13) and (2.14):

$$(E')^{2j+1} = (F')^{2j+1} = \mathbf{1} \tag{3.8}$$

$$(E')^\beta (F')^\alpha = (F')^\alpha (E')^\beta \omega(-\alpha\beta). \tag{3.9}$$

It is easy to show that these transformations form a group which is the $SL(2, Z(2j+1))$ group. These transformations are the analogue of the $SL(2, R)$ transformations in the harmonic oscillator phase space, which lead to the Bogoliubov transformations:

$$\begin{aligned} &x' = \lambda x + kp \qquad && p' = \mu x + \nu p \\ &\lambda, k, \mu, \nu \in \mathbb{R} \qquad && \lambda\nu - \mu k = 1. \end{aligned} \tag{3.10}$$

In the Galois case (equation (3.2)) for any α, β, γ there always exists a δ which satisfies equation (3.7):

$$\delta = (1+\beta\gamma)a^{-1}. \tag{3.11}$$

In the non-Galois case (equation (3.1)), only for some triplets α, β, γ does there exist a δ which satisfies equation (3.7). Of course, a non-Galois system can be factorized in terms of Galois subsystems (equation (3.5)) and $SL(2, Z(p_\ell^{n_\ell}))$ transformations can be performed on each Galois subsystem.

In [5] we have constructed explicitly important special cases of the $SL(2, Z(2j+1))$ transformations. Here we construct explicitly the operators corresponding to general $SL(2, Z(2j+1))$ transformations for Galois angular momentum systems. We first consider the unitary operator

$$T = \sum_{m=-j}^{j} \omega(2^{-1}m^2)|J; jm\rangle\langle J; jm| \tag{3.12}$$

$$TT^+ = T^+T = \mathbf{1}. \tag{3.13}$$

The powers of this operator

$$T^\lambda = \sum_{m=-j}^{j} \omega(2^{-1}\lambda m^2)|J; jm\rangle\langle J; jm| \tag{3.14}$$

form a subgroup of $SL(2, Z(2j+1))$. It is easy to show

$$T^\lambda E^\beta T^{-\lambda} = E^\beta F^{\lambda\beta}\omega(2^{-1}\lambda\beta) \tag{3.15}$$

$$[T^\lambda, F] = 0. \tag{3.16}$$

We now show that an arbitrary element of $SL(2, Z(2j+1))$ can be written as

$$U(\nu, \mu, \lambda) = T^\nu U_F T^\mu U_F T^\lambda. \tag{3.17}$$

Using equations (2.27), (3.15) and (3.10) we show

$$E' \equiv U(\nu, \mu, \lambda)EU^+(\nu, \mu, \lambda) = E^\alpha F^\beta \omega(\varepsilon) \tag{3.18}$$

$$F' \equiv U(\nu, \mu, \lambda)FU^+(\nu, \mu, \lambda) = E^\gamma F^\delta \omega(\eta) \tag{3.19}$$

where

$$\alpha = \lambda\mu - 1 \qquad \beta = \nu\lambda\mu - \nu - \lambda$$
$$\gamma = \mu \qquad \delta = (1+\beta\gamma)\alpha^{-1} = \mu\nu - 1 \tag{3.20}$$

and

$$\varepsilon = 2^{-1}(\nu\lambda\mu - \lambda\mu - \nu + \lambda) - \lambda^2\mu \qquad \eta = 2^{-1}(\mu\nu - 3\mu). \tag{3.21}$$

The transformations (3.18), (3.19) are indeed the same as the transformations (3.6) with the extra phase factors $\omega(\varepsilon)$, $\omega(\eta)$. We can easily solve equation (3.20) so that for given α, β, γ (with $\alpha \neq 0$, $\gamma \neq 0$) the corresponding λ, μ, ν are

$$\mu = \gamma \qquad \lambda = (1+\alpha)\gamma^{-1} \qquad \nu = (\beta\gamma + \alpha + 1)(\alpha\gamma)^{-1}. \tag{3.22}$$

Note that the inverses do exist in the Galois case studied here. In the case $\gamma = 0$, equation (3.7) gives $\alpha\delta = 1 \pmod{2j+1}$ and the transformations (3.6) reduce to

$$E \to E^\alpha \qquad F \to F^{\alpha^{-1}}. \tag{3.23}$$

This special case has been considered in [5] where we have shown that the operators

$$R(\alpha) = \sum_{n=-j}^{j} |J; j\,\alpha n\rangle\langle J; jn| \tag{3.24}$$

give

$$R(a)ER^+(\alpha) = E^\alpha \tag{3.25}$$

$$R(\alpha)FR^+(\alpha) = F^{\alpha^{-1}}. \tag{3.26}$$

In the case $\alpha = 0$, equation (3.7) gives $\beta\gamma = -1$ and the transformations (3.6) reduce to

$$E \to F^{\beta} \tag{3.27}$$

$$F \to E^{-\beta^{-1}}. \tag{3.28}$$

Clearly in this case the operators

$$S(\beta) = R(\beta^{-1})U_F \tag{3.29}$$

give

$$S(\beta)ES^{+}(\beta) = F^{\beta} \tag{3.30}$$

$$S(\beta)FS^{+}(\beta) = E^{-\beta^{-1}}. \tag{3.31}$$

We have, therefore, constructed explicitly operators that perform the transformations (3.6) in the general case. Clearly these operators can be constructed in many different ways and here we have presented one of them.

Acting with these operators on the J and θ states and operators we get J' and θ' states and operators along different lines in the phase space:

$$\begin{aligned} &UJ_iU^{+} \equiv J_i' \qquad U\theta_iU^{+} \equiv \theta_i' \qquad i = +, -, z \\ &U|J;\, jm\rangle \equiv |J';\, jm\rangle \qquad U|\theta;\, jm\rangle \equiv |\theta';\, jm\rangle. \end{aligned} \tag{3.32}$$

They are the Bogoliubov transformations in a discrete context. Note that the phase space $Z(2j+1) \times Z(2j+1)$ is a set of points which form a finite geometry [10] only in the Galois case in which $(2j+1)$ is a power of a prime. In this case it makes sense to talk about lines and many other geometrical properties. In the non-Galois case the phase space is just a set of points with no geometrical structure whatsoever. This also indicates why in the Galois case we get stronger results. The harmonic oscillator phase space methods rely on the existence of a classical phase space (a plane) that has a geometrical structure. In the finite quantum systems studied here, only in the Galois case do we have a phase space with geometrical structure and, as a result of this, we can prove stronger results than in the non-Galois case.

4. Displacement operators and their properties in the Galois case

We have already introduced displacement operators in section 2 and studied many of their properties. Here we give some stronger results which are valid in the Galois case only. We use the notation

$$D(\alpha, \beta) = F^{\alpha}E^{\beta}\omega(-2^{-1}\alpha\beta) \tag{4.1}$$

$$D(\alpha_1, \beta_1)D(\alpha_2, \beta_2) = D(\alpha_1+\alpha_2, \beta_1+\beta_2)\omega(2^{-1}\alpha_1\beta_2 - 2^{-1}\alpha_2\beta_1) \tag{4.2}$$

$$D(0,0) = D(2j+1, 0) = D(0, 2j+1) = D(2j+1, 2j+1) = \mathbf{1}. \tag{4.3}$$

The 2^{-1} is the inverse of 2 and is an integer which always exists in the Galois case considered here. Note that the displacement operators (4.1) do not have an arbitrary phase factor but a particular one ($\omega(-2^{-1}\alpha\beta)$). This is crucial for the properties we are going to prove here. These properties are

$$(2j+1)^{-1}\sum_{\beta=-j}^{j} D(\alpha, \beta) = |\theta;\, j\, 2^{-1}\alpha\rangle\langle\theta;\, j\, -2^{-1}\alpha| \tag{4.4}$$

$$(2j+1)^{-1}\sum_{\alpha=-j}^{j} D(\alpha, \beta) = |J;\, j\, 2^{-1}\beta\rangle\langle J;\, j\, -2^{-1}\beta|. \tag{4.5}$$

In order to prove equation (4.4) we first show that

$$(2j+1)^{-1}\sum_{\beta=-j}^{j}E^{\beta}\omega(-2^{-1}\alpha\beta)=|\theta;\,j-2^{-1}\alpha\rangle\langle\theta;\,j-2^{-1}\alpha|. \qquad (4.6)$$

Indeed, taking the matrix elements of both sides of equation (4.6) with $\langle\theta;\,jm|$ and $|\theta;\,jn\rangle$ and using equation (2.16) we get

$$\delta(m,n)(2j+1)^{-1}\sum_{\beta=-j}^{j}\omega(-\beta n-2^{-1}\alpha\beta)=\delta(m,-2^{-1}\alpha)\delta(n,-2^{-1}\alpha) \qquad (4.7)$$

where $\delta(m,n)$ are Kronecker deltas. Equation (4.7) can be proved using the results from the appendix (equation (A6)). We now act with F^{α} on the left of both sides of equation (4.6) and, using equation (2.18), we obtain equation (4.4). In a similar way we can prove equation (4.5).

We next introduce the parity operator

$$P_0=U_F^2=\sum_{m=-j}^{j}|\theta;\,j-m\rangle\langle\theta;\,jm|=\sum_{m=-j}^{j}|J;\,j-m\rangle\langle J;\,jm| \qquad (4.8)$$

$$P_0^2=\mathbf{1}. \qquad (4.9)$$

We easily see that

$$(2j+1)^{-1}\sum_{\alpha,\beta}D(\alpha,\beta)=P_0 \qquad (4.10)$$

and that

$$D(\alpha,\beta)P_0=P_0D(-\alpha,-\beta). \qquad (4.11)$$

The displaced parity operator has been studied in the harmonic oscillator context in [11]. It has been shown that it is intimately connected to the Wigner function. Its properties are similar to those of the Wigner function and its trace with respect to the density matrix of the system gives the Wigner function. In the present context the displaced parity operator is defined as

$$P(\alpha,\beta)=D(\alpha,\beta)P_0D^{+}(\alpha,\beta)=D(2\alpha,2\beta)P_0=P_0D^{+}(2\alpha,2\beta) \qquad (4.12)$$

where the equalities are proved with the use of equation (4.11). Acting with P_0 on the right of both sides of equation (4.4), (4.5) and (4.10) we get

$$(2j+1)^{-1}\sum_{\beta=-j}^{j}P(\alpha,\beta)=|\theta;\,j\alpha\rangle\langle\theta;\,j\alpha| \qquad (4.13)$$

$$(2j+1)^{-1}\sum_{\alpha=-j}^{j}P(\alpha,\beta)=|J;\,j\beta\rangle\langle J;\,j\beta| \qquad (4.14)$$

$$(2j+1)^{-1}\sum_{\alpha,\beta}P(\alpha,\beta)=\mathbf{1}. \qquad (4.15)$$

If we now define the Wigner function as

$$W(\alpha,\beta)=\mathrm{Tr}[\rho P(\alpha,\beta)] \qquad (4.16)$$

where ρ is the density matrix of the system ($\mathrm{Tr}\,\rho = 1$), we can show through equations (4.13), (4.14) and (4.15) that

$$(2j+1)^{-1}\sum_{\beta=-j}^{j} W(\alpha,\beta) = \langle\theta;\, j\alpha|\rho|\theta;\, j\alpha\rangle \tag{4.17}$$

$$(2j+1)^{-1}\sum_{\alpha=-j}^{j} W(\alpha,\beta) = \langle J;\, j\beta|\rho|J;\, j\beta\rangle \tag{4.18}$$

$$(2j+1)^{-1}\sum_{\alpha,\beta} W(\alpha,\beta) = 1. \tag{4.19}$$

There is already an extensive literature on the Wigner functions of finite systems (e.g. [12]). Our intention here is not to study them as a problem in their own right, but to show the connection between their properties and the properties (4.4), (4.5) and (4.10) of the displacement operators.

We also point out that acting with the operators U and U^{+}, that perform $SL(2, Z(2j+1))$ transformations on the left and right, correspondingly, of equation (4.13), and using equation (3.32) we obtain a similar equation to (4.13) but on a different line in the finite phase space. The ability to perform these transformations is a powerful result which, as we explained at the end of section 3 is valid only in the Galois case.

5. Central extensions of G by $Z(2j+1)$

We consider the group $\mathcal{D}$ with elements

$$D(\alpha,\beta,\gamma) = F^{\alpha}E^{\beta}\omega(\gamma). \tag{5.1}$$

These are the displacement operators of the previous section (but with an arbitrary phase factor $\omega(\gamma)$ where γ belongs to $Z(2j+1)$). It is clear that

$$D(\alpha_1,\beta_1,\gamma_1)D(\alpha_2,\beta_2,\gamma_2) = D(\alpha_2,\beta_2,\gamma_2)D(\alpha_1,\beta_1,\gamma_1)\omega(\alpha_1\beta_2-\alpha_2\beta_1). \tag{5.2}$$

Equation (5.2) can also be expressed in terms of the commutator of $D(\alpha_1,\beta_1,\gamma_1)$ and $D(\alpha_2,\beta_2,\gamma_2)$ as

$$\begin{aligned}[][D(\alpha_1,\beta_1,\gamma_1), D(\alpha_2,\beta_2,\gamma_2)] &\equiv [D(\alpha_1,\beta_1,\gamma_1)]^{-1}[D(\alpha_2,\beta_2,\gamma_2)]^{-1}\\ &\quad\times D(\alpha_1,\beta_1,\gamma_1)D(\alpha_2,\beta_2,\gamma_2) = \omega(\alpha_1\beta_2-\alpha_2\beta_1).\end{aligned} \tag{5.3}$$

We also consider the Abelian group $G = \mathcal{D}|Z(2j+1)$ with elements which are the cosets

$$\begin{aligned} g(\alpha,\beta) &= \{D(\alpha,\beta,\gamma)|\ \text{arbitrary}\ \gamma\ \text{in}\ Z(2j+1)\}\\ g(\alpha_1,\beta_1)g(\alpha_2,\beta_2) &= g(\alpha_1+\alpha_2,\beta_1+\beta_2).\end{aligned} \tag{5.4}$$

The non-commutativity between F^{α} and E^{β}, as a result of which we get the phase factor $\omega(\alpha_1\beta_2-\alpha_2\beta_1)$ in equation (5.2), is essential for quantum mechanics. Here we consider the question of whether there are many ways of constructing the group $\mathcal{D}$ from the Abelian group G and $Z(2j+1)$. Mathematically this corresponds to studying the central extensions of G by $Z(2j+1)$.

We consider elements

$$D(\alpha,\beta,\gamma) = g(\alpha,\beta)\omega(\gamma) \tag{5.5}$$

with the multiplication rule

$$D(\alpha_1, \beta_1, \gamma_1)D(\alpha_1, \beta_2, \gamma_2) = [g(\alpha_1, \beta_1)\omega(\gamma_1)][g(\alpha_2, \beta_2)\omega(\gamma_2)]$$
$$= g(\alpha_1 + \alpha_2, \beta_1 + \beta_2)\omega(\gamma_1 + \gamma_2 + \sigma(\alpha_1, \beta_1; \alpha_2, \beta_2)). \quad (5.6)$$

The $\sigma(\alpha_1, \beta_1; \alpha_2, \beta_2)$ is called a factor set and is an arbitrary real function restricted by the associativity rule which implies

$$\sigma(\alpha_1, \beta_1; \alpha_2, \beta_2) + \sigma(\alpha_1 + \alpha_2, \beta_1 + \beta_2; \alpha_3, \beta_3)$$
$$= \sigma(\alpha_1, \beta_1; \alpha_2 + \alpha_3, \beta_2 + \beta_3) + \sigma(\alpha_2, \beta_2; \alpha_3, \beta_3) \quad (5.7)$$

and also by

$$\sigma(0, 0; \alpha, \beta) = \sigma(\alpha, \beta; 0, 0) = 0. \quad (5.8)$$

We call $\mathcal{D}(\sigma)$ the group related to a given σ. The function $\sigma(\alpha_1, \beta_1; \alpha_2, \beta_2)$ is a two-cocycle. The definition of a two-cocycle is

$$\delta\sigma = \sigma(\alpha_1, \beta_1; \alpha_2\beta_2) + \sigma(\alpha_1 + \alpha_2, \beta_1 + \beta_2; \alpha_3, \beta_3)$$
$$-\sigma(\alpha_1, \beta_1; \alpha_2 + \alpha_3, \beta_2 + \beta_3) - \sigma(\alpha_2, \beta_2; \alpha_3, \beta_3) = 0 \quad (5.9)$$

and is precisely the associativity requirement (5.7). Here δ is the coboundary operator with the property

$$\delta^2 = 0. \quad (5.10)$$

We call $Z^2(G, Z(2j + 1))$ the group of two-cocycles.

Let $\tau(\alpha, \beta)$ be an arbitrary function such that

$$\tau(0, 0) = 0. \quad (5.11)$$

Then

$$\sigma(\alpha_1, \beta_1; \alpha_2, \beta_2) = \tau(\alpha_1, \beta_1) + \tau(\alpha_2, \beta_2) - \tau(\alpha_1 + \alpha_2, \beta_1 + \beta_2) \quad (5.12)$$

obeys (5.7), (5.8) and is a special case of a two-cocycle. It is by definition a two-coboundary:

$$\delta\tau = \tau(\alpha_1, \beta_1) + \tau(\alpha_2, \beta_2) - \tau(\alpha_1 + \alpha_2, \beta_1 + \beta_2). \quad (5.13)$$

We call $B^2(G, Z(2j + 1))$ the group of two-coboundaries. The two-cohomology group is

$$H^2(G, Z(2j + 1)) = Z^2(G, Z(2j + 1))|B^2(G, Z(2j + 1)). \quad (5.14)$$

Each factor $\sigma(\alpha_1, \beta_1; \alpha_2, \beta_2)$ (defined modulo $\tau(\alpha_1, \beta_1) + \tau(\alpha_1, \beta_2) - \tau(\alpha_1 + \alpha_2; \beta_1 + \beta_2)$) characterizes a two-cohomology class. More specifically, since the two-coboundary of equation (5.13) is symmetric under transformations $(\alpha_1, \beta_1) \leftrightarrow (\alpha_2, \beta_2)$, it is the antisymmetric part of $\sigma(\alpha_1, \beta_1; \alpha_2, \beta_2)$ that characterizes the cohomology class.

We next consider the commutator of two elements of $\mathcal{D}(\sigma)$,

$$[D(\alpha_1, \beta_1, \gamma_1), D(\alpha_2, \beta_2, \gamma_2)] \equiv [D(\alpha_2, \beta_2, \gamma_2)]^{-1}$$
$$\times D(\alpha_1, \beta_1, \gamma_1)]^{-1}[D(\alpha_1, \beta_1, \gamma_1)D(\alpha_2, \beta_2, \gamma_2)$$
$$= \omega[A(\alpha_1, \beta_1; \alpha_2, \beta_2)] \quad (5.15)$$

where

$$A(\alpha_1, \beta_1; \alpha_2, \beta_2) = -A(\alpha_2, \beta_2; \alpha_1, \beta_1) = \sigma(\alpha_1, \beta_1; \alpha_2, \beta_2) - \sigma(\alpha_2, \beta_2; \alpha_1, \beta_1) \quad (5.16)$$

is the antisymmetric part of $\sigma(\alpha_1, \beta_1; \alpha_2, \beta_2)$. It is known [6] that for central extensions

$$[D(\alpha_1, \beta_1, \gamma_1)D(\alpha_2, \beta_2, \gamma_2), D(\alpha_3, \beta_3, \gamma_3)]$$
$$= [D(\alpha_1, \beta_1, \gamma_1), D(\alpha_3, \beta_3, \gamma_3)][D(\alpha_2, \beta_2, \gamma_2), D(\alpha_3, \beta_3, \gamma_3)] \quad (5.17)$$

and this implies

$$A(\alpha_1+\alpha_2, \beta_1+\beta_2; \alpha_3, \beta_3) = A(\alpha_1, \beta_1; \alpha_3, \beta_3) + A(\alpha_2, \beta_2; \alpha_3, \beta_3). \quad (5.18)$$

Using equations (5.8), (5.16) we prove

$$A(0, 0; \alpha, \beta) = A(\alpha, \beta; 0, 0) = 0. \quad (5.19)$$

Looking at the above properties of $A(\alpha_1, \beta_1; \alpha_2, \beta_2)$ we conclude that it is a multiple of $\alpha_1\beta_2 - \alpha_2\beta_1$:

$$A(\alpha_1, \beta_1; \alpha_2, \beta_2) = \nu(\alpha_1\beta_2 - \alpha_2\beta_1). \quad (5.20)$$

Similar results in a different context have been derived in [7]. Here, however, the required periodicity leads to the quantization of ν. More specifically we require that for $\alpha_1 = 2j+1$, $\beta_1 = 0$, $\alpha_2 = 0$, $\beta_2 = 1$ the commutator of equation (5.15) is equal to one and, therefore, $A(2j+1, 0; 0, 1)$ takes the values $N(2j+1)$ where N is an integer. Therefore the antisymmetric part of $\sigma(\alpha_1, \beta_1; \alpha_2, \beta_2)$ is

$$A(\alpha_1, \beta_1; \alpha_2, \beta_2) = N(\alpha_1\beta_2 - \alpha_2\beta_1). \quad (5.21)$$

As we explained above, this antisymmetric part characterizes the cohomology class to which $\sigma(\alpha_1, \beta_1; \alpha_2, \beta_2)$ belongs. Therefore the cohomology classes are labelled by the integer N. It is now clear that

$$\sigma(\alpha_1, \beta_1; \alpha_2, \beta_2) = N(\alpha_1\beta_2 - \alpha_2\beta_1) + \tau(\alpha_1, \beta_2) + \tau(\alpha_2, \beta_2) - \tau(\alpha_1+\alpha_2, \beta_1+\beta_2). \quad (5.22)$$

The 'τ-terms' can be viewed as 'gauge transformations'. If σ_1, σ_2 are characterized by the same N and different $\tau_1(\alpha, \beta)$, $\tau_2(\alpha, \beta)$ and if the $D_1(\alpha, \beta, \gamma)$ are elements of $\mathcal{D}(\sigma_1)$ (i.e. multiply according to the rule σ_1), it is easy to see that

$$D_2(\alpha, \beta, \gamma) \equiv D_1(\alpha, \beta, \gamma) \exp\{\mathrm{i}[\tau_2(\alpha, \beta) - \tau_1(\alpha, \beta)]\} \quad (5.23)$$

are elements of $\mathcal{D}(\sigma_2)$ (i.e. multiply according to the rule σ_2).

The integer N characterizes the strength of the non-commutativity. The compactness of the model leads to the result that N is an integer. Note that in non-compact models (e.g. [7]) the analogous parameter is not an integer and is interpreted as one of the constants in the system (e.g. mass, etc).

6. Discussion and application to other areas

We have considered the angle-angular momentum quantum phase space and studied the Heisenberg–Weyl group. When $Z(2j+1)$ is a Galois field, due to the fact that every non-zero element has an inverse, stronger results can be proved. We have constructed explicitly in equation (3.17) general $SL(2, Z(2j+1))$ transformations for Galois systems in terms of the operators T of equation (3.12) and the Fourier operators U_F. We have also studied the properties of the displacement operators and proved equations (4.4), (4.5) and (4.10). In section 5 we have explored the most general way of constructing the Heisenberg–Weyl group from the Abelian group $G = \mathcal{D}/Z(2j+1)$ and $Z(2j+1)$. We have shown that there exist many cohomology classes labelled by an integer N.

The above ideas have been presented in the context of the angle-angular momentum but they are also relevant in other contexts. In [13] the displacement operators $D(\alpha, \beta)$ of equation (4.1) are used as generators for the $SU(2j+1)$ transformations in the Hilbert space $H(2j+1)$. Equation (4.2) leads to the commutator (a different quantity from the commutator of equation (5.3)):

$$
\begin{aligned}
[D(\alpha_1, \beta_1), D(\alpha_2, \beta_2)] &\equiv D(\alpha_1, \beta_1)D(\alpha_2, \beta_2) - D(\alpha_2, \beta_2)D(\alpha_1, \beta_1) \\
&= 2\mathrm{i}\sin\left[\frac{2\pi}{2j+1}2^{-1}(\alpha_1\beta_2 - \alpha_2\beta_1)\right] D(\alpha_1+\alpha_2, \beta_1+\beta_2).
\end{aligned}
\tag{6.1}
$$

Equations (6.1) form the algebra for $SU(2j+1)$. Of special interest in this context is the limit $j \to \infty$ in which we get an infinite-dimensional algebra related to area-preserving diffeomorphisms. In this context we have shown in [5] that we can get a ladder of angle operators θ_{jm} (the lowest of which are the angle operators θ_+, θ_-, θ_z used here), in complete analogy to the known ladder of angular momentum operators J_{jm} (the lowest of which are the angular momentum operators J_+, J_-, J_z).

Other areas where the ideas considered in this paper can be applied are the magnetic translation group for two-dimensional electron systems in magnetic fields [14]; the quantum Hall effect (e.g. [15]); hydrodynamics [16]; etc. Work on coherent states in truncated finite Hilbert spaces [17] is also related to this topic. More recently, [18] have discussed the practical implementation of the transformations in finite quantum systems, with beam splitters in optical systems. The Galois field aspects studied in this paper could be useful in implementing coding theory ideas in this context. It is, therefore, clear that there is a very wide range of potential applications of the ideas expressed in this paper.

Appendix

We introduce here the d_m functions which are the analogues of the delta function and its derivatives in the harmonic oscillator case. The d_m functions can be useful in practical calculations of matrix elements. We start with the function

$$d_0(x) = (2j+1)^{-1}\sum_{\ell=-j}^{j}\omega(\ell x) \tag{A1}$$

$$d_0(x+2j+1) = d_0(x). \tag{A2}$$

For $x = 0$ (modulo $2j+1$) we obtain

$$d_0(0) = 1. \tag{A3}$$

For non-zero x we easily prove

$$d_0(x) = (2j+1)^{-1}\frac{\sin(\pi x)}{\sin(\pi x/(2j+1))}. \tag{A4}$$

We see that

$$d_0(x) = d_0(-x). \tag{A5}$$

When x takes integer values, $d_0(x)$ is one if x is equal to zero (modulo $2j+1$) and zero for all other integer values of x,

$$d_0(n) = \delta(n, 0) \tag{A6}$$

where $\delta(n, 0)$ is the Kronecker delta (it is equal to 1 when n is equal to 0 modulo $(2j+1)$).

We now introduce the function

$$d_m(x) = (2j+1)^{-1}\sum_{\ell=-j}^{j}\left(\mathrm{i}\frac{2\pi\ell}{2j+1}\right)^m\omega(\ell x) = \partial_x^m d_0(x) \tag{A7}$$

$$d_m(x+2j+1) = d_m(x). \tag{A8}$$

Using equation (A5) we prove

$$d_m(-x) = (-1)^m d_m(x). \tag{A9}$$

For $x = 0$ we get

$$d_m(0) = (2j+1)^{-1} \sum_{\ell=-j}^{j} \left(\mathrm{i}\frac{2\pi\ell}{2j+1}\right)^m \tag{A10}$$

and we easily see that for odd m the result is zero,

$$d_{2k+1}(0) = 0. \tag{A11}$$

For even m the result is different from zero. We give the result for the first few even values of m:

$$d_2(0) = -\frac{j(j+1)}{3}\left(\frac{2\pi}{2j+1}\right)^2 \tag{A12}$$

$$d_4(0) = \frac{1}{15} j(j+1)(3j^2+3j-1)\left(\frac{2\pi}{2j+1}\right)^4 \tag{A13}$$

$$d_6(0) = \frac{1}{15} j(j+1)(3j^4+6j^3-3j+1)\left(\frac{2\pi}{2j+1}\right)^6. \tag{A14}$$

Equation (A7) shows that $d_m(x)$ is the finite Fourier transform of ℓ^m, a similar result to the continuous case where $\delta^{(m)}(x)$ is the Fourier transform of x^m. Equation (A7) simplifies as follows:

$$m = 4k \rightarrow d_m(x) = (2j+1)^{-1} \sum_{\ell=1}^{j} 2\left(\frac{2\pi\ell}{2j+1}\right)^m \cos\left[\frac{2\pi}{2j+1}\ell x\right] \tag{A15}$$

$$m = 4k+1 \rightarrow d_m(x) = -(2j+1)^{-1} \sum_{\ell=1}^{j} 2\left(\frac{2\pi\ell}{2j+1}\right)^m \sin\left[\frac{2\pi}{2j+1}\ell x\right] \tag{A16}$$

$$m = 4k+2 \rightarrow d_m(x) = -(2j+1)^{-1} \sum_{\ell=1}^{j} 2\left(\frac{2\pi\ell}{2j+1}\ell x\right)^m \cos\left[\frac{2\pi}{2j+1}\ell x\right] \tag{A17}$$

$$m = 4k+3 \rightarrow d_m(x) = (2j+1)^{-1} \sum_{\ell=1}^{j} 2\left(\frac{2\pi\ell}{2j+1}\right)^m \sin\left[\frac{2\pi}{2j+1}\ell x\right]. \tag{A18}$$

We now prove that for a positive integer n

$$\langle J;\, jk|\theta_z^n|J;\, j\ell\rangle = (2j+1)^{-1} \sum_{m=-j}^{j} m^n \omega(m(k-\ell)) = \left(-\mathrm{i}\frac{2j+1}{2\pi}\right)^n d_n(k-\ell) \tag{A19}$$

$$\langle \theta;\, jk|J_z^n|\theta;\, j\ell\rangle = (2j+1)^{-1} \sum_{m=-j}^{j} m^n \omega(m(\ell-k)) = \left(-\mathrm{i}\frac{2j+1}{2\pi}\right)^n d_n(\ell-k). \tag{A20}$$

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